GOSTS Descriptive Set Theory

Properties of Sets of Reals

James Holland

2020-10-28

The Plan

- Introduce three commonly studied properties a set of reals can have.
  1. Perfect, κ-Suslin sets
  2. Lebesgue measurability
  3. The Baire Property
- All Borel sets will have these properties.
- $\utilde{\mathbf{\Sigma}}^1_1$ sets will too, but this is where things get hairy.
- Mostly the problem with trying to go beyond $\utilde{\mathbf{\Sigma}}^1_1$ is because L has some strong regularity properties like a $\utilde{\mathbf{\Delta}}^1_2$ well-ordering of $\mathcal{N}$.

Perfect Sets

- Our study of perfect sets will be the most involved of the three properties.
- We're going to have to introduce more technology and rely on more set theory than with the Baire property or Lebesgue measurability.

Definition

Let $\underset{\sim}{\mathcal{M}}$ be a topological space.

- A point x in a subset $X \subseteq \mathcal{M}$ is *isolated* iff some open U has $U \cap X = \{x\}$.
- A set $X \subseteq \mathcal{M}$ is *perfect* iff it is closed and has no isolated points.

- If the topological space has no isolated points, it's perfect.
- E.g. $\mathcal{N}$, $\mathcal{C}$, $\mathbb{R}$ are all perfect.
- $[a, b] \subseteq \mathbb{R}$ is perfect for all $a < b \in \mathbb{R}$.
- $[0, 1] \cup \{-4, 5\}$ is not perfect because -4 and 5 are both isolated.
- A countable set is never perfect.
  - (The only way for a countable set to have no isolated points is for it to look like $\mathbb{Q}$ which isn't closed.)

Perfect Sets

Definition

A set X is *perfect* iff it is closed and has no isolated points.

Perfect sets are important because in polish spaces, every closed set is only countably many points away from being perfect.

Theorem (Cantor–Bendixson)

Every closed subset $A \subseteq \mathcal{N}$ is either countable or contains a perfect subset. (And in fact can be uniquely decomposed into $A = X \cup Y$ for perfect X and countable Y.)

Proof.

- Let $A \subseteq \mathcal{N}$ be closed but not perfect.
- Note that $A \setminus \{x \in A : x \text{ is isolated}\}$ is still closed.
- So we just keep removing the isolated points (just countably many times by separability) until we're left with either $\emptyset$ (in which case A was countable) or some non-empty, closed X.
- This X will be perfect and $Y = A \setminus X$ will be countable. ⊣

Perfect Sets

Note that we can't just remove the set of isolated points from $A \in \underset{\sim}{\mathbf{\Pi}}^0_1$ and call it a day. A might be countable, but not all of its elements isolated:

$$A = \{0, 1/n : n \in \omega\} \subseteq \mathbb{R}$$
$$\{x \in A : x \text{ is isolated}\} = \{1/n : n \in \omega\} \subsetneq A.$$

So we have to remove the isolated points twice to arrive at $\emptyset$.

$$A = \{0, 1/n + 1/m : n, m \in \omega\} \subseteq \mathbb{R}$$

Requires us to remove the isolated points three times to arrive at $\emptyset$.

Nevertheless, with a countable set, we always arrive at $\emptyset$ in countably many stages.

Perfect Sets

This gives another look into what the closed sets of a polish space look like: continuous copies of Cantor space.

Lemma

Let $\underset{\sim}{\mathcal{M}}$ be a polish space. Let $f : \mathcal{C} \to \mathcal{M}$ be a continuous injection. Therefore $\operatorname{im} f$ *is perfect in $\underset{\sim}{\mathcal{M}}$.*

Proof.

- $\operatorname{im} f$ has no isolated points since $\{f(x)\}$ being open implies $\{x\}$ is open which doesn't happen in $\underset{\sim}{\mathcal{C}}$.
- To show $\operatorname{im} f$ is closed, suppose $\langle f(x_i) : i < \omega \rangle$ converges in $\underset{\sim}{\mathcal{M}}$ to y.
- If $\langle x_i : i < \omega \rangle$ converges to x in $\underset{\sim}{\mathcal{C}}$, then by continuity,

$$y = \lim_{i \to \infty} f(x_i) = f(\lim_{i \to \infty} x_i) = f(x) \in \operatorname{im} f.$$

- By compactness of $\underset{\sim}{\mathcal{C}}$, we get a subsequence that converges. So $\operatorname{im} f$ is closed. ⊣

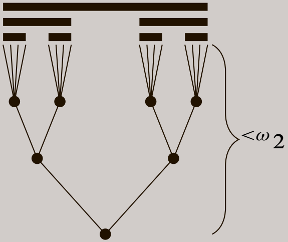
Perfect Sets

We also get a partial converse. (17 A • 5 in the notes.)

Lemma

Let $\underset{\sim}{\mathcal{M}}$ be a polish space. Let $P \subseteq \mathcal{M}$ be perfect. Therefore there is a perfect subset $C \subseteq P$ that (with the inherited topology) is homeomorphic to $\underset{\sim}{\mathcal{C}}$.

Note every perfect set will be homeomorphic to $\underset{\sim}{\mathcal{C}}$ since, e.g., $\mathbb{R}$ is perfect but not homeomorphic to $\underset{\sim}{\mathcal{C}}$. Nevertheless, $\mathbb{R}$ has a subset that is: the Cantor set.



The path along an $x \in {}^{\omega}2$ corresponds to deciding which interval you're in after you've taken out the middle third.

The Perfect Set Property

All of this is just to motivate the perfect set property (PSP).

Definition

A set X has $\mathsf{PSP}(X)$ iff X is countable or else there is a perfect $P \subseteq X$. A pointclass $\utilde{\Gamma}$ has $\mathsf{PSP}(\utilde{\Gamma})$ iff every $X \in \utilde{\Gamma}$ has $\mathsf{PSP}(X)$.

Hence $\mathsf{PSP}(\utilde{\Pi}^0_1)$.

- This really just to allow us to encompass countable sets and non-closed sets.
- Question: is there a set without the perfect set property?

The next theorem tells us that any such set will need to be very complicated because it won't even be Borel.

Theorem

Every Borel subset of a polish space has PSP.

This is one use of changing the topology.

The Perfect Set Property

Theorem

PSP(X) *for every Borel X in any polish space $\underset{\sim}{\mathcal{M}}$.*

Proof.

- For $X \subseteq \mathcal{M}$ uncountable and Borel, there is a topology $\underset{\sim}{\mathcal{M}}'$ where X is clopen and $\underset{\sim}{\Sigma}_1^{0,\underset{\sim}{\mathcal{M}}} \subseteq \underset{\sim}{\Sigma}_1^{0,\underset{\sim}{\mathcal{M}}'}$.
- Since X is closed there, there is a perfect (in $\underset{\sim}{\mathcal{M}}'$) $P \subseteq X$.
- WLOG, $P = \operatorname{im} f$ for an injective $f : \mathcal{C} \to X$ continuous (as a function from $\underset{\sim}{\mathcal{C}}$ to $\underset{\sim}{\mathcal{M}}'$)
- Then f is continuous from $\underset{\sim}{\mathcal{C}}$ to $\underset{\sim}{\mathcal{M}}$ since $\underset{\sim}{\mathcal{M}}$ has fewer open sets than $\underset{\sim}{\mathcal{M}}'$.
- Hence $P \subseteq X$ is perfect in $\underset{\sim}{\mathcal{M}}$ so PSP(X). $\dashv$

The Perfect Set Property

This begs the question: what *doesn't* have PSP?

- Note that $\mathsf{PSP}(X)$ says either $|X| \leq \aleph_0$ or else $|X| = 2^{\aleph_0}$.
- In particular X isn't a counterexample to CH (that every set of reals has size $\leq |\mathbb{N}|$ or $|\mathbb{R}|$).
- This provides some motivation why the attempt to construct a counterexample to CH is fruitless: such a counterexample will be very complicated because "simple" sets have PSP.
- Nevertheless, we *can* find a set without PSP because of AC.

Result

There is a set $X \subseteq \mathcal{N}$ such that $\neg\mathsf{PSP}(X)$.

- The idea is to well-order the perfect sets and diagonalize.
- Proof is 17 A • 8 in the notes.

The Perfect Set Property

- In general, the set we get isn't going to fit into our hierarchy, relying so much on AC. But it does demonstrate the limits to PSP.
- In particular, in **L** we will be able to find a $\underset{\sim}{\mathbf{\Pi}}^1_1$ set without the perfect set property, sort-of due to the definable well-ordering of the reals of L (cf. Kanamori, Theorem 13.12)
- That said, such a set might *actually* have PSP, it's just that L doesn't have the relevant functions (a continuous injection $f : \mathcal{C} \to X$ or a surjection $f : \omega \to X$).
- Thus the best we most we could hope for in general is $\mathsf{PSP}(\underset{\sim}{\mathbf{\Sigma}}^1_1)$.
- It turns out this is exactly what we get: $\mathsf{ZFC} \vdash \mathsf{PSP}(\underset{\sim}{\mathbf{\Sigma}}^1_1)$.
- That said, under stronger (unprovable) large cargdinal hypotheses, we get $\mathsf{PSP}(\underset{\sim}{\mathbf{\Sigma}}^1_n)$ for $n > 1$.
- In particular, if n Woodin cardinals exist, then $\mathsf{PSP}(\underset{\sim}{\mathbf{\Sigma}}^1_{n+1})$ holds.
- Such results are related to determinacy, as $\mathsf{Det}(X)$ implies $\mathsf{PSP}(X)$.

The remainder of our discussion on PSP will be more-or-less about showing PSP($\underset{\sim}{\Sigma}^1_1$).

Definition

A tree over ω is *perfect* iff it always eventually splits (i.e. $\tau \in T$ extends to two $\sigma_1, \sigma_2 \in T$ on different branches).

Result

$X \subseteq \mathcal{N}$ is perfect iff $X = [T]$ for some perfect tree T over ω.

Proof.

- Infinite branches of trees are always closed.
- That it's perfect is't so difficult: that it always eventually splits bascially means we can identify a subset of T with ${}^{<\omega}2$.
- (Full proof is 17 A • 10 in the notes.) ⊣

Suslin Sets

We have that $\mathbf{\Sigma}^1_1$-sets, being projections of closed sets, have the form $\mathfrak{p}[T]$ for T a tree over $\omega \times \omega$. Let us generalize this.

Definition

For $X \subseteq \mathcal{N}$ and α an ordinal, X is α-Suslin iff $X = \mathfrak{p}_{\mathcal{N}}[T]$ for some tree T over $\omega \times \alpha$.

- Here we really regard elements of T as pairs of finite sequences (of the same length).
- Branches of T similarly are regarded as pairs of infinite sequences (in ${}^\omega\omega \times {}^\omega\alpha = \mathcal{N} \times {}^\omega\alpha$).

Result

$X \subseteq \mathcal{N}$ is $\mathbf{\Sigma}^1_1$ iff X is $\aleph_0$-Suslin.

Suslin Sets

More generally, we have the following.

Result

$X \subseteq \mathcal{N}$ is $|X|$-Suslin. In particular, every subset of $\mathcal{N}$ is $2^{\aleph_0}$-Suslin.

Proof.

Let $f : X \to |X|$ be a bijection. Consider the tree (visually a bunch of separate lines branching off of $\langle \emptyset, \emptyset \rangle$)

$$T = \{\langle x \restriction n, \mathrm{const}_{f(x)} \restriction n \rangle \in {}^n\omega \times {}^n|X| : n < \omega \wedge x \in X\}.$$

T is a tree over $\omega \times |X|$ with $X \subseteq \mathfrak{p}[T]$. To see $\mathfrak{p}[T] \subseteq X$, any branch of T is the form $\langle x, \mathrm{const}_{f(x)} \rangle$ for $x \in \mathcal{N}$, $\alpha < |X|$ so that $\alpha = f(x)$ and so $x \in X$. ⊣

As an exercise, see what goes wrong if we don't properly consider the side conditions and just take the tree

$$T' = \{\langle x \restriction n, \mathrm{const}_0 \restriction n \rangle : n < \omega\}.$$

Suslin Sets

Result

$X \subseteq \mathcal{N}$ is $|X|$-Suslin.

This result is fine, but it doesn't give us explicit ways of forming the tree: they rely on this bijection which ignores the complexity of X.

- We are more interested in complexity and the αs which the sets are α-Suslin for.
- We often want definability restrictions on the sorts of trees we care about.

We can show, for example, that $\utilde{\boldsymbol{\Pi}}^1_1$-sets are all $\aleph_1$-Suslin (which trivially holds if CH is true since then all sets are $2^{\aleph_0} = \aleph_1$-Suslin).

Suslin Sets

This result is especially important because it introduces the "Shoenfield tree".

Result

$X \in \underset{\sim}{\boldsymbol{\Pi}}^1_1$ *implies* X *is* $\aleph_1$*-Suslin.*

Proof.

- If $X \in \underset{\sim}{\boldsymbol{\Pi}}^1_1$, then its complement is $\aleph_0$-Suslin: $X = \mathfrak{p}[T]$ for some tree over $\omega \times \omega$.
- This means $x \in X$ iff there is *no* infinite branch $\langle x, y \rangle \in [T]$.
- $T_x = \{\tau \in {}^n\omega : n < \omega \wedge \langle x \restriction n, \tau \rangle \in T\}$ is the tree of things corresponding to x.
- $x \in X$ iff there is no infinite branch of T_x: $[T_x] = \emptyset$.
- If we turn T_x "upside down", this means $\langle T_x, \triangleright \rangle$ is a well-founded partial order (there are no infinite descending sequences).
- Well-founded relations can be identified with rank functions: $\sigma \mathrel{R} \tau \rightarrow \text{rank}(\sigma) < \text{rank}(\tau)$ (or $\sigma \triangleright \tau \rightarrow \text{rank}(\sigma) < \text{rank}(\tau)$ in our case.)

Suslin Sets

Result

$X \in \utilde{\Pi}^1_1$ implies X is $\aleph_1$-Suslin.

Proof.

- $x \in X$ iff $[T_x] = \emptyset$ iff $\langle T_x, \rhd \rangle$ is well-founded iff there is a rank function $\text{rank} : T_x \to \omega_1$.
- Consider the tree of approximations to such rank functions: $\langle \tau, R \rangle \in S_T \subseteq {}^{<\omega} \times {}^{<\omega}\omega_1$ iff for all $\sigma^*, \sigma < \text{lh}(\tau)$,
  1. If $\sigma^* \rhd \sigma$,
  2. and if $\langle \tau \restriction \text{lh}(\sigma^*), \sigma^* \rangle \in T$; then
  3. $R(\sigma^*) < R(\sigma)$.
- Basically, R is a rank function on T_τ up to the length of τ.

Suslin Sets

Result

$X \in \underset{\sim}{\Pi}^1_1$ implies X is $\aleph_1$-Suslin.

Proof.

- Basically, for $\langle \tau, R \rangle \in S_T$, R is a rank function on T_τ up to the length of τ.
- The infinite branches of S_T are of the form $\langle x, R \rangle$ where R is a rank function on T_x.
- Hence $x \in X$ iff $[T_x] = \emptyset$ iff there is a rank function on T_x iff $\langle x, \text{rank} \rangle \in [S_T]$ iff $x \in \mathfrak{p}[S_T]$.
- So S_T as a tree over $\omega \times \omega_1$ yields X as ω_1-Suslin. ⊣

Some nice, standard corollaries of this are the following:

- $\underset{\sim}{\Pi}^1_1$-sets are unions of $\aleph_1$-many Borel sets (17 A • 18)
- every $\underset{\sim}{\Pi}^1_1$-set is either countable, of size $\aleph_1$, or of size $2^{\aleph_0}$ (17 A • 19)

Suslin Sets

So how will we show PSP($\underset{\sim}{\Sigma}^1_1$)? I'll give a *sketch* of the long, technical proof of a more general result (17 A • 20 in the notes).

Result

Let $X \subseteq \mathcal{N}$ be κ-Suslin. Therefore $|X| > \kappa$ implies X contains a perfect subset.

Proof.

- Let T witness X is κ-Suslin. We proceed similarly to the Cantor–Bendixson analysis before: keep removing isolated points.
- In terms of trees, this means if $\langle \tau, r \rangle \in T$, we remove it if there aren't extensions on different branches.
- The resulting tree, prune(T) might have *introduced* isolated branches, but we can keep pruning and eventually this will stabilize at some stage $\alpha < \kappa^+$. This gives $T^* \subseteq T$
- If we have pruned everything away ($T^* = \emptyset$) then $|X| \leq \kappa$ because we only remove $\leq \kappa$-many elements at each stage.
- If $T^* \neq \emptyset$, we can build a copy of ${}^{<\omega}2$ in T^*. This gives a perfect tree T' over ω with $[T'] \subseteq \mathfrak{p}[T^*] = X$. ⊣

Suslin Sets

Result

Let $X \subseteq \mathcal{N}$ be κ-Suslin. Therefore $|X| > \kappa$ implies $\mathsf{PSP}(X)$.

- So since $\mathbf{\Sigma}^1_1$-sets consist precisely of the $\aleph_0$-Suslin sets, any $X \in \mathbf{\Sigma}^1_1$ is either countable or contains a perfect subset, i.e. $\mathsf{PSP}(X)$.
- It turns out that, similar to $\mathbf{\Pi}^1_1$-sets, $\mathbf{\Sigma}^1_2$-sets are $\aleph_1$-Suslin.
- There's much more to say about this material, but not that I can fit in this already overstuffed series.
- This concludes the discussion on perfect sets, by far the most technical and set-theoretic of the three properties.
- So if you're asleep at this point, you can start paying attention again.

Lebesgue Measure

Of the three properties we care about,

- the most well known is being Lebesgue measurable.
- the hardest to define is being Lebesgue measurable.

I'll skip the tedious detail of actually defining Lebesgue measure (μ) on $\mathbb{R}$. From here, we can identify Lebesgue measure on other spaces.

Theorem (Measure Isomorphism Theorem)

Let $\underset{\sim}{\mathcal{M}}$ be an uncountable polish space.
Let $\mu_{\mathcal{M}}$ be a non-trivial probability measure on $\underset{\sim}{\mathcal{M}}$ over at least the Borel sets of $\underset{\sim}{\mathcal{M}}$.
Therefore there is a bijection $f : [0,1] \to \mathcal{M}$ such that

1. *$X \subseteq [0,1]$ is Borel iff $f"X \subseteq \mathcal{M}$ is Borel;*
2. *$X \subseteq [0,1]$ is Lebesgue measurable iff $f"X \subseteq \mathcal{M}$ is $\mu_{\mathcal{M}}$-measurable; and*
3. *$\mu(X) = \mu_{\mathcal{M}}(f"X)$ for all Lebesgue measurable X.*

In short, there is a well-defined class of Lebesgue measurable sets across any (uncountable) Polish space (modulo translations by Borel isomorphisms).

Lebesgue Measure

By closure properties of the projective pointclasses, questions of which projective pointclasses consist of lebesgue measurable sets have the same answer across all (uncountable) polish spaces.

We *can* define a measure over $\mathcal{N}$, but it's not exactly intuitive: $\mu^*(\mathcal{N}_\tau) = \prod_{n<\mathrm{lh}(\tau)} \frac{1}{2^{\tau(n)+1}}$ and we extend to an outer measure from here by covering sets with disjoint cones.

A basic overview of general measure theory stuff: working over polish spaces, we get the following. (They're not really relevant here, but worth knowing.)

Result

- *For any outer measure μ^*, μ^* is a measure on the σ-algebra of μ^*-measurable sets;*
- *For any measure μ on a σ-algebra Σ, we get an outer measure μ^* such that $\mu^* \restriction \Sigma = \mu$;*
- *And in this case, the set of μ^*-measurable sets is the largest σ-algebra over which μ^* is a measure.*

Nice Properties of Lebesgue Measure

More important for us is that the Lebesgue measurable sets consist precisely of the Borel sets modulo null sets.

Result

The family of lebesgue measurable sets is $\{B \cup N : B \text{ is Borel} \wedge N \text{ is null}\} \supseteq \mathcal{B}$.

Less commonly known about Lebesgue measure is the existence of a "minimal" measurable set for any (potentially non-measurable) set.

Lemma

For $X \subseteq \mathbb{R}$, there's a measurable $A \supseteq X$ that's "minimal" in that if $X \subseteq A' \subseteq A$ with A' measurable, then $A \setminus A'$ is null.

Proof.

- For $\varepsilon > 0$, let $U_\varepsilon \supseteq X$ be open where $\mu^*(U_\varepsilon)$ is within ε of $\mu^*(X)$.
- $A = \bigcap_{n\in\omega} U_{1/n}$ is Borel with $\mu^*(A) = \mu^*(X)$.
- If $X \subseteq A' \subseteq A$, $\mu^*(A') = \mu^*(A)$ so by measurability, $\mu^*(A \setminus A') = 0$. ⊣

Nice Properties of Lebesgue Measure

- So which sets are lebesgue measurable mostly comes down to thinking about what sets are lebesgue null.
- It's a classic result that not every $X \subseteq \mathbb{R}$ is lebesgue measurable via a *Vitali* set.
- The construction of such a set depends on a given choice function.
- Given that $\mathcal{N}^{\mathsf{L}}$ has a $\utilde{\mathbf{\Delta}}^1_2$-well-ordering, this means the Vitali set constructed is actually projective (and $\utilde{\mathbf{\Delta}}^1_2$) and hence $\mathsf{L} \models$ "not every $\utilde{\mathbf{\Delta}}^1_2$-set is measurable".
- With large cardinal axioms (incompatible with "V = L") we get the consistency of more and more of the projective hierarchy being measurable:
  - a measurable cardinal implies $\utilde{\mathbf{\Sigma}}^1_2$-sets are measurable,
  - an inaccessible implies the consistency of $\utilde{\mathbf{\Sigma}}^1_3$-sets being measurable,
  - ω Woodin cardinals implies all projective sets are measurable.
- Such ideas are (again) related to determinacy.
- This suggests the best we could show in ZFC alone is that $\utilde{\mathbf{\Sigma}}^1_1$-sets are measurable, and this is precisely what we get.

The Suslin Operation

To show $\underset{\sim}{\Sigma}^1_1$-sets are measurable, we need a way of generating $\underset{\sim}{\Sigma}^1_1$-sets from closed sets that preserves lebesgue measurability.

Definition

Consider a family indexed by finite sequence of ω: $X = \{X_\tau : \tau \in^{<\omega} \omega\}$. The *suslin operation* applied to X is $\mathcal{A}X = \bigcup_{x \in \mathcal{N}} \bigcap_{n \in \omega} X_{x \restriction n}$.

- Note that $\mathcal{A}X$ depends on the association $\tau \mapsto X_\tau$ rather than just X itself.
- Some history: $\underset{\sim}{\Sigma}^1_1$-sets are sometimes call *analytic*.
- This seems to come from Luzin and these sets being called $\mathcal{A}$-sets.
- The '$\mathcal{A}$' itself seems to have been proposed by Suslin in honor of (Pavel) Aleksandrov.
- I avoid using "analytic" because it's too close to the analytical hierarchy, a separate but related hierarchy we'll discuss later.

Lemma

A set is $\underset{\sim}{\Sigma}^1_1$ iff it is $\mathcal{A}X$ for some $X = \{X_\tau : \tau \in {}^{<\omega}\omega\} \subseteq \underset{\sim}{\Pi}^0_1$.

The Suslin Operation

Lemma (17 B • 18)

A set is $\boldsymbol{\Sigma}^1_1$ iff it is $\mathcal{A}X$ for some $X = \{X_\tau : \tau \in {}^{<\omega}\omega\} \subseteq \boldsymbol{\Pi}^0_1$.

Proof.

Just use trees over $\omega \times \omega$ and project down:

($\leftarrow$) Each $X_\tau = [T_\tau]$, T_τ a tree over ω. Consider

$$T = \left\{\langle \tau, \sigma\rangle : \mathrm{lh}(\tau) = \mathrm{lh}(\sigma) \wedge \tau \in \bigcap_{n<\mathrm{lh}(\sigma)} T_{\sigma\restriction n}\right\}$$

It turns out $\langle x, y\rangle \in [T]$ iff $x \in [T_{y\restriction n}]$ for every $n < \omega$ iff $x \in \bigcap_{n<\omega} X_{y\restriction n}$. So $\mathcal{A}X = \mathfrak{p}[T]$.

($\rightarrow$) $X \in \boldsymbol{\Sigma}^1_1$ is $\mathfrak{p}[T]$ for some T over $\omega \times \omega$. Define

$$T_\sigma = \{\tau \in {}^{<\omega}\omega : \exists\sigma' \text{ comparable with } \sigma\ (\langle \tau, \sigma'\rangle \in T)\}.$$

Each $[T_\sigma] \in \boldsymbol{\Pi}^0_1$, and it turns out $X = \mathcal{A}\{[T_\sigma] : \sigma \in {}^{<\omega}\omega\}$. ⊣

The Suslin Operation

It turns out that the lebesgue measurable sets are closed under the suslin operation $\mathcal{A}$ and as $\mathbf{\Pi}^0_1$-sets are measurable, so too are all $\mathbf{\Sigma}^1_1$-sets.

Lemma

Let $X = \{X_\tau : \tau \in {}^{<\omega}\omega\}$ be a family of measurable sets. Therefore $\mathcal{A}X$ is measurable.

Proof.

- WLOG $X_\tau \subseteq X_\sigma$ for $\sigma \trianglelefteq \tau$ (replacing X_τ with $\bigcap_{\sigma \trianglelefteq \tau} X_\sigma$).
- Define $A_{\geq\tau} = \bigcup_{x \in \mathcal{N}_\tau} \bigcap_{n\in\omega} X_{x \restriction n}$ so $\mathcal{A}X = A_{\geq\emptyset}$.
- Each $A_{\geq\sigma} \subseteq X_\sigma$. Get minimal measurable sets $B_{\geq\sigma}$ where (intersecting with X_σ) $A_{\geq\sigma} \subseteq B_{\geq\sigma} \subseteq X_\sigma$.
- It follows that $N_\tau = B_{\geq\tau} \setminus \bigcup_{n<\omega} B_{\geq\tau^\frown\langle n\rangle}$ is null.
- So the union of them is null and so $B = B_{\geq\emptyset} \setminus \bigcup_{\tau\in{}^{<\omega}\omega} N_\tau$ is measurable.
- (With some work) we get $B \subseteq \mathcal{A}X$ and $\mathcal{A}X \setminus B \subseteq \bigcup_{\tau\in{}^{<\omega}\omega} N_\tau$ is null so that $\mathcal{A}X$ is measurable. $\dashv$

The Baire Property

The last idea is more topological involving less analysis of $\mathbb{R}$ and more about polish spaces. The motivating fact is the following theorem similar to forcing.
(Density for $\underset{\sim}{\mathcal{N}}$ means any $\tau \in {}^{<\omega}\omega$ can be extended to a real in the set.)

Theorem (Baire Category Theorem)

Let $\{D_n : n < \omega\}$ be a collection of open dense subsets of $\underset{\sim}{\mathcal{N}}$. Therefore $\bigcap_{n<\omega} D_n \neq \emptyset$.

Proof.

- The intersection of any two dense open sets is dense and open.
- Hence we can build up a real by intersecting each D_n: $\tau_0 \in D_0$;
- For τ_n defined, by density there's an extension $\tau_n \trianglelefteq x \in \bigcap_{i \leq n+1} D_i$. By openness, x has a neighborhood $\mathcal{N}_\tau \subseteq \bigcap_{i \leq n+1} D_i$.
- Taking τ large enough so $\tau_n \trianglelefteq \tau$ gives $\tau_{n+1} = \tau$.
- $\bigcup_{n<\omega} \tau_n \in \mathcal{N}$ and in $\bigcap_{n<\omega} D_n$. $\dashv$

The Baire Property

The Baire property uses a sense of smallness similar to Lebesgue measure, but instead of thinking about how the set can be covered, we care about how dense the set is.

Definition

- A set X is *nowhere dense* iff $\mathcal{N} \setminus X$ *contains* a dense open set.
- A set X is *meagre* iff it's the countable union of nowhere dense sets.
- A set X has the *Baire property* ($\mathsf{BP}(X)$) iff $X \triangle U$ is meagre for some open U.

So $\emptyset \neq \bigcap_{n<\omega} D_n$ for open dense D_n implies $\mathcal{N} \neq \bigcup_{n<\omega} X_n$ where each X_n is nowhere dense. In particular, $\mathcal{N}$ isn't meagre.

There are a *ton* of examples of this sort of thing: (17 C • 4) in *any* topological space $\underset{\sim}{\mathcal{M}}$,

- For any open U, $\mathrm{cl}(U) \setminus U$ is nowhere dense.
- If X is nowhere dense and $Y \subseteq X$ then Y is nowhere dense.
- If X is meagre and $Y \subseteq X$ then Y is meagre.
- The finite union of nowhere dense sets is nowhere dense.
- The countable union of meagre sets is meagre.

The Baire Property

To motivate the terminology a bit, we have the following.

Result

For $\mathcal{M}$ a topological space, $X \subseteq \mathcal{M}$ is nowhere dense *iff for every open $U \subseteq \mathcal{M}$, $X \cap U$ is not dense (in the inherited topology on U).*

Proof.

($\rightarrow$) If $X \cap U$ is dense in U then $X \cap W \neq \emptyset$ for every open $W \subseteq U$.

- Let $D \subseteq \mathcal{M} \setminus X$ be open, dense so $U \setminus D$ is closed in U containing the dense set X.
- The only closed dense sets are U itself: $U \setminus D = U$ contradicting density of D in U.

($\leftarrow$) If $X \cap U$ is not dense for every open $U \subseteq \mathcal{M}$, then every (non-empty) open $U \subseteq \mathcal{M}$ has a (non-empty) open $W \subseteq U$ with $W \cap X = \emptyset$. So consider $D = \bigcup\{W \text{ open} : W \cap X = \emptyset\}$. It follows that D is dense and open. $\dashv$

The Baire Property

Getting back on track, the main route for showing $\underset{\sim}{\Sigma}^1_1$-sets all have the Baire property is the following:

Result (17 C • 5)

In any topological space, the family of sets with the BP *is a σ-algebra trivially containing the open sets and so* $\mathsf{BP}(X)$ *for every* $X \in \mathcal{B}$.

We also get a similar "minimality" result as with lebesgue measure.

Lemma (17 C • 8)

For every $X \subseteq \mathcal{N}$, *there is a* $B \supseteq X$ *such that* $\mathsf{BP}(B)$ *and if* $X \subseteq B' \subseteq B$ *then* $B \setminus B'$ *is meagre whenever* $\mathsf{BP}(B')$.

From here, we can basically copy and paste the proof of the closure of the lebesgue measurable sets under $\mathcal{A}$, replacing "null" with "meagre".

Theorem (17 C • 9)

Let $X = \{X_\tau : \tau \in {}^{<\omega}\omega\}$ *be such that* $\mathsf{BP}(X_\tau)$ *for each* $\tau \in {}^{<\omega}\omega$. *Therefore* $\mathsf{BP}(\mathcal{A}X)$.

The Baire Property

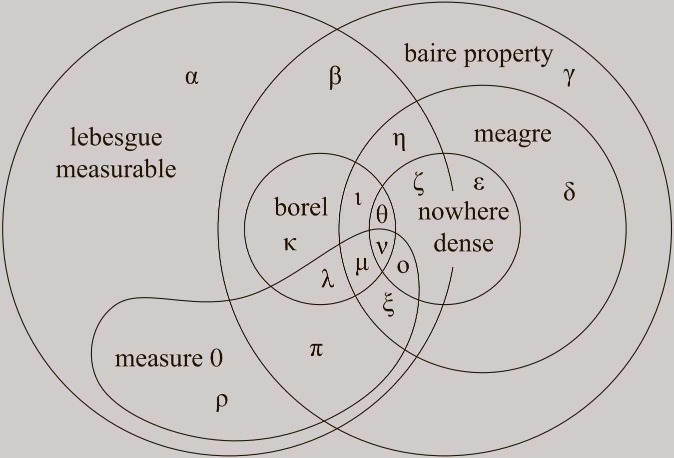
Corollary

Every $\underset{\sim}{\Sigma}^1_1$-set has the Baire property.

- That entire sketch is basically just to show that arguments involving BP act similarly as with lebesgue measurability.
- As usual, not every set has the Baire property (e.g. any Bernstein set).
- And L thinks there's a $\underset{\sim}{\Delta}^1_2$-set without BP.
- Nevertheless, the two concepts are completely different.

The Baire Property

In ZFC, all of the intersections here are non-empty in non-trivial ways:



The Baire Property

- let B be any bernstein set (intersects every uncountable closed set but contains none of them);
- let Non be the set of non-normal numbers in [0, 1];
- let C be the cantor set and FC the fat cantor set (any version); and
- let NBC be any non-borel subset of C.

α. $(\mathrm{B} \cap \mathrm{Non}) \cup [1,2]$;

β. $\mathrm{NBC} \cup [1,2]$;

γ. $\mathrm{B} \cup \mathrm{Non}$;

δ. $[0,1] \cap \mathrm{B} \setminus \mathrm{Non}$

ε. $\mathrm{FC} \cap \mathrm{B}$;

ζ. $\mathrm{FC} \setminus \mathrm{NBC}$;

η. $\mathrm{NBC} \cup \mathbb{Q} \cup [1,2]$;

θ. FC;

ι. $\mathrm{FC} \cup \mathbb{Q}$;

κ. $\mathbb{R}$;

λ. An $N \in \underset{\sim}{\mathbf{\Pi}}^0_2$ where $\mathbb{R} = M \cup N$ with M meagre and N null;

μ. $\mathbb{Q}$;

ν. C;

ξ. $\mathrm{NBC} \cup \mathbb{Q}$;

ο. NBC;

π. Non;

ρ. $\mathrm{B} \cap \mathrm{Non}$.